Section 2.3

1. (a) Find the number of positive integer divisors of $642 = 2^3 3^4$.
   (b) Find the sum of all positive integer divisors of 642.

Solution

(a) Each divisor must have the form $2^i 3^j$ where $0 \leq i \leq 3$ and $0 \leq j \leq 4$. Hence, there are $3 \cdot 4 = 12$ divisors of 642.

(b) The sum of the divisors is

$$\begin{aligned}
\sum_{i=0}^{3}\sum_{j=0}^{4} 2^i 3^j &= 2^0 \sum_{j=0}^{4} 3^j + 2^1 \sum_{j=0}^{4} 3^j + 2^2 \sum_{j=0}^{4} 3^j + 2^3 \sum_{j=0}^{4} 3^j && (1)\\
&= (2^0 + 2^1 + 2^2 + 2^3)121 && (2)\\
&= 1815. && (3)
\end{aligned}$$

2. Find a formula for the sum of all divisors of integers of the form $2^m 3^n$ $(m, n \geq 0)$.

Solution Each divisor must have the form $2^i 3^j$ where $0 \leq i \leq m$ and $0 \leq j \leq n$. Hence, the sum of the divisors is

$$\begin{aligned}
\sum_{i=0}^{m}\sum_{j=0}^{n} 2^i 3^j &= 2^0 \sum_{j=0}^{n} 3^j + 2^1 \sum_{j=0}^{n} 3^j + \cdots + 2^m \sum_{j=0}^{n} 3^j && (4)\\
&= (2^0 + 2^1 + 2^2 + \cdots + 2^m) \sum_{j=0}^{n} 3^j && (5)\\
&= (2^{m+1} - 1)\frac{3^{n+1} - 1}{2}. && (6)
\end{aligned}$$

$\star$3. (Problem A1 from the 1989 William Lowell Putnam Mathematics Competition.) Consider the following sequence of integers (in base 10): $101, 10101, 1010101, 101010101, 10101010101, \ldots$. Prove that 101 is the only number in this sequence that is prime. (*Hint*: Use place value to write each number in terms of the sum of its digits; for example, $abcde = a10^4 + b10^3 + c10^2 + d10 + e$. Then examine how the sum might be factored.)

Solution It is easily checked that 101 is prime. Given any number of the form $10101\ldots01$ greater than 101, there is an integer $n \geq 2$ such that

$$\begin{aligned} 10101\ldots01 &= 10^{2n} + 10^{2n-2} + \cdots + 10^4 + 10^2 + 1 && (7)\\ &= \frac{10^{2n+2} - 1}{99} && (8)\\ &= \frac{(10^{n+1})^2 - 1}{99} && (9)\\ &= \frac{(10^{n+1} - 1)(10^{n+1} + 1)}{99} && (10)\\ &= \frac{a_n(10^{n+1} + 1)}{11} && (11) \end{aligned}$$

where a_n is the integer that is a string of $n + 1$ 1s. If n is odd, then $11|a_n$. If n is even, then $11|(10^{n+1}+1)$. In either case, $10101\ldots01$ is a product of two integers, each greater than 1. Therefore $10101\ldots01$ is not prime if $n > 1$.